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Orthogonal Expansion of Real Polynomials, Location of Zeros, and an L^2 Inequality

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Let $f(z)=a_0\phi_0(z)+a_1\phi_1(z)+\cdots+a_n\phi_n(z)$ be a polynomial of degree n, given as an orthogonal expansion with real coefficients. We study the location of the zeros of f relative to an interval and in terms of some of the coefficients. Our main theorem generalizes or refines results due to Turán and Specht. In particular, it includes a best possible criterion for the occurrence of real zeros. Our approach also allows us to establish a weighted L^2 inequality giving a lower estimate for the product of two polynomials.

Key Words: orthogonal expansion; real polynomials; location of zeros; criterion for real zeros; weighted L^2 inequality.

1. INTRODUCTION AND STATEMENT OF RESULTS

Turán [17] proposed to study the zeros of a polynomial f in terms of the coefficients $a_0, a_1, \ldots, a_n$ of an orthogonal expansion

$$f(z)=a_0\phi_0(z)+a_1\phi_1(z)+\cdots+a_n\phi_n(z).$$

He himself [17–19] obtained various results for the Hermite expansion with

$$\phi_\nu(z)=H_\nu(z):=(-1)^\nu\,\mathrm{e}^{z^2}\frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}(\mathrm{e}^{-z^2})\qquad(\nu\in\mathbb{N}_0). \tag{1}$$

Similar results for general orthogonal expansions were obtained by Specht [12–15] and subsequently by numerous other authors; for further references see [9].

For basic facts about orthogonal polynomials, we refer to [3, 5, 16]. Throughout this paper, we shall use the following notations. By σ, we denote

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an m-distribution, that is, a non-decreasing bounded function $\sigma\colon \mathbb{R} \to \mathbb{R}$ which attains infinitely many distinct values and is such that the moments

$$\mu_n := \int_{-\infty}^{\infty} x^n \, d\sigma(x) \qquad (n \in \mathbb{N}_0)$$

exist. Then there exists a uniquely determined sequence of polynomials

$$\phi_0(z), \phi_1(z), ..., \phi_n(z), ..., \tag{2}$$

called the *sequence of monic orthogonal polynomials with respect to* $d\sigma(x)$, with the following properties:

(i) each ϕ_ν $(\nu \in \mathbb{N}_0)$ is a monic polynomial of degree ν;

(ii) $\int_{-\infty}^{\infty} \phi_n(x)\, \phi_m(x)\, d\sigma(x) = 0$ for $n \neq m$.

For an arbitrary polynomial f, we write

$$\|f\| := \left(\int_{-\infty}^{\infty} |f(x)|^2 \, d\sigma(x) \right)^{1/2} \tag{3}$$

and introduce the numbers

$$\gamma_n := \|\phi_n\|^2 \qquad (n \in \mathbb{N}_0). \tag{4}$$

The zeros of ϕ_n are known to be real. By J_n, we denote the smallest interval containing the zeros of ϕ_n and define a distance function d_n by

$$d_n(z) := \min\{ |z - \zeta| : \zeta \in J_n \} \qquad (z \in \mathbb{C}). \tag{5}$$

Since the zeros of consecutive orthogonal polynomials interlace, we have

$$d_1(z) \geqslant d_2(z) \cdots \geqslant d_n(z) \geqslant \cdots \geqslant |\Im z|.$$

A typical result, which may be attributed to Specht (see Giroux [4] who proves a refinement), is as follows.

THEOREM A. *Let*

$$f(z) = a_0 \phi_0(z) + a_1 \phi_1(z) + \cdots + a_n \phi_n(z)$$

be a polynomial of degree n, given as an orthogonal expansion with complex coefficients. Then, in the above notations, each zero ζ *of f satisfies the inequality*

$$d_n(\zeta) \leqslant \sqrt{\sum_{\nu=0}^{n-1} \frac{\gamma_\nu}{\gamma_{n-1}} \left| \frac{a_\nu}{a_n} \right|^2}. \tag{6}$$

It is desirable to have additional and refined results for polynomials which are real-valued on the real line. In the following, the numbers

$$C_{n,k} := \frac{1}{\gamma_k} \max\{\|\psi_k\|^2 : \psi_k \text{ a monic divisor of } \phi_n \text{ with } \deg \psi_k = k\} \tag{7}$$

will be of significance.

Let $\xi_1, ..., \xi_n$ denote the zeros of ϕ_n. It can be shown by calculating $\|\psi_{n-1}\|^2$ with the help of the Gaussian quadrature formula with nodes $\xi_1, ..., \xi_n$ that

$$C_{n,n-1} = \max_{1 \leqslant v \leqslant n} \frac{\phi_n'(\xi_v)}{\phi_{n-1}(\xi_v)}.$$

In the case that $d\sigma(x)$ has bounded support, a simple upper bound for $C_{n,k}$ was mentioned in [10]. In [9], we presented the following theorem.

THEOREM B. *Let*

$$f(z) = a_0\phi_0(z) + a_1\phi_1(z) + \cdots + a_n\phi_n(z) \tag{8}$$

be a polynomial of degree n, given as an orthogonal expansion with real coefficients. Then in the above notations, each zero ζ of f satisfies the inequality

$$|\Im\zeta| \leqslant \left(\sqrt{\sum_{v=0}^{n-2} \frac{\gamma_v}{\gamma_{n-2}} \left|\frac{a_v}{a_n}\right|^2} - \frac{\gamma_{n-1}}{\gamma_{n-2}\sqrt{C_{n-1,n-2}}}\right)^{1/2} \tag{9}$$

provided that the term in parentheses is non-negative, else f has n distinct real zeros which separate those of ϕ_{n-1}.

The criterion for real zeros contained in Theorem B is best possible in the sense that $C_{n-1,n-2}$ cannot be replaced by a smaller number (see Theorem 1 below).

Theorem B includes a quantitative version of the fact [16, Theorem 3.3.4] that an orthogonal binomial

$$a_{n-1}\phi_{n-1}(z) + a_n\phi_n(z)$$

with real coefficients which do not both vanish has simple real zeros, and thus, if the coefficients of the polynomial (8) are real and $a_0, ..., a_{n-2}$ are of sufficiently small modulus, then f has also simple real zeros. Note that the coefficient a_{n-1} does not appear on the right-hand side of (9). Another result involving only some of the coefficients was proved in [10, Theorem 1]. Following a common convention, we denote by $\lfloor x \rfloor$ the largest integer not exceeding x.

THEOREM C. *Let*

$$f(z)=a_0\phi_0(z)+a_1\phi_1(z)+\cdots+a_n\phi_n(z) \tag{10}$$

be a polynomial given as an orthogonal expansion with real coefficients. Let k be an integer with $0\leqslant k\leqslant n$, and set $m:=\lfloor (n+k+2)/2\rfloor$. If, in terms of the numbers (4) *and* (7),

$$\sum_{j=0}^{k-1}\gamma_j a_j^2<\frac{\gamma_k}{C_{m,k}-1}\,a_k^2,$$

then f has at least k distinct real zeros of odd multiplicities, lying in the smallest interval spanned by the zeros of ϕ_m.

In Section 2, we shall show that in Theorem C the constant $C_{m,k}$ cannot be replaced by a smaller number (see the remark following the proof of Lemma 3).

Theorem C includes a quantitative version of the fact [1, 8] that a polynomial

$$a_k\phi_k(z)+a_{k+1}\phi_{k+1}(z)+\cdots+a_n\phi_n(z)\qquad (a_k\neq 0)$$

with real coefficients $a_k, ..., a_n$ has at least k distinct real zeros odd multiplicities, and thus, if the polynomial (10) has real coefficients and $a_0, ..., a_{k-1}$ are of relatively small modulus as compared to $|a_k|$, then f has also at least k distinct real zeros of odd multiplicities. Moreover, there may be seen a relationship to results of van Vleck [20], Montel [7], and Ballieu [2], who showed for a polynomial $f(z)=\sum_{\nu=0}^{n} b_\nu z^\nu$ that the coefficients $b_0, ..., b_{k-1}$ along with one further coefficient $b_m\neq 0$ $(1\leqslant k\leqslant m\leqslant n)$ allow to construct a bound for k of the zeros of f. Theorem C would correspond to the case that $m=k$.

Our main result is an analogue for an orthogonal expansion in the case that $m=n$. It will generalize and refine Theorem B. Let us first introduce the functions

$$K_{\mu,\nu}\colon x\mapsto\begin{cases}\dfrac{1}{\sqrt{C_{\mu,\nu}}}\left(x-\dfrac{\gamma_\mu}{\gamma_\nu\sqrt{C_{\mu,\nu}}}\right) & \text{if}\quad 0\leqslant x<\dfrac{2\gamma_\mu}{\gamma_\nu\sqrt{C_{\mu,\nu}}}\\[3ex] \dfrac{\gamma_\nu}{4\gamma_\mu}\,x^2 & \text{if}\quad \dfrac{2\gamma_\mu}{\gamma_\nu\sqrt{C_{\mu,\nu}}}\leqslant x<\dfrac{2\gamma_\mu}{\gamma_\nu}\\[3ex] x-\dfrac{\gamma_\mu}{\gamma_\nu} & \text{if}\quad x\geqslant\dfrac{2\gamma_\mu}{\gamma_\nu}\end{cases} \tag{11}$$

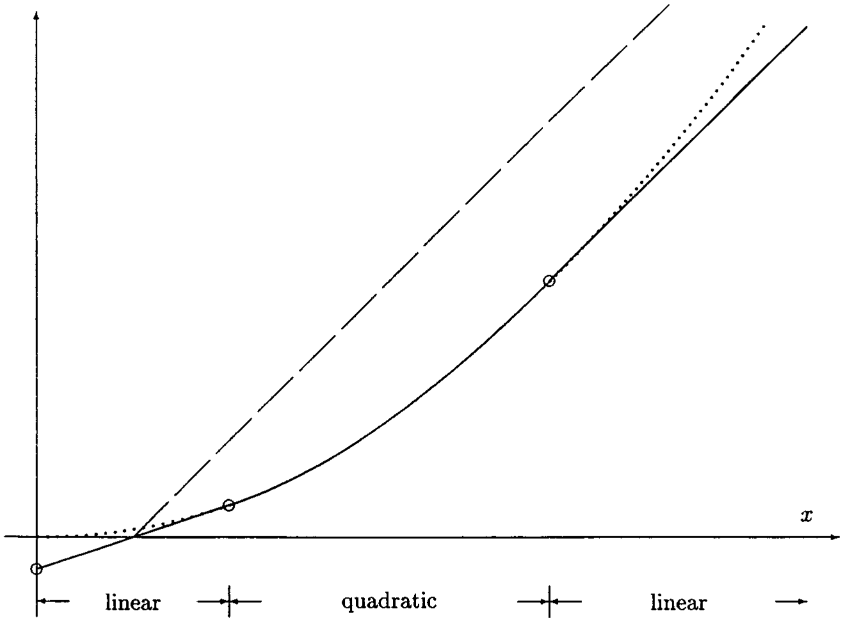

FIG. 1. The graph of the function $K_{\mu,\nu}$, a quadratic majorant as a dotted curve, and a linear majorant for positive $K_{\mu,\nu}(x)$ as a dashed line.

($\mu, \nu \in \mathbb{N}_0$, $\mu \geqslant \nu$), which are easily seen to be monotonically increasing and continuously differentiable (cf. Fig. 1).

THEOREM 1. *Let*

$$f(z) = a_0\phi_0(z) + a_1\phi_1(z) + \cdots + a_n\phi_n(z)$$

be a polynomial of degree $n \geqslant 2$, given as an orthogonal expansion with real coefficients. Let k be an integer such that $2 \leqslant k \leqslant n$ and $n-k$ is even, and set $m := (n+k)/2 - 1$. If, in terms of the numbers (4) *and* (7),

$$\sum_{\nu=0}^{k-2} \gamma_\nu a_\nu^2 < \frac{\gamma_m^2}{\gamma_{k-2} C_{m,k-2}} a_n^2, \tag{12}$$

then f has at least k distinct real zeros of odd multiplicities, else f has at most $(n-k)/2$ pairs of conjugate zeros in the region

$$\left\{ z \in \mathbb{C} : d_m(z) > \left[K_{m,k-2}\left(\sqrt{\sum_{\nu=0}^{k-2} \frac{\gamma_\nu}{\gamma_{k-2}} \left| \frac{a_\nu}{a_n} \right|^2} \right) \right]^{1/(n-k+2)} \right\}$$

with the function $K_{m,k-2}$ given by (11). *In* (12) *the constant $C_{m,k-2}$ cannot be replaced by a smaller number.*

Since the function $K_{\mu,\nu}$ is defined piecewise, one may like to replace it by a linear or a quadratic majorant. It is geometrically evident (cf. Fig. 1) that

$$K_{\mu,\nu}(x) \leqslant x - \frac{\gamma_\mu}{\gamma_\nu \sqrt{C_{\mu,\nu}}} \qquad \text{for} \quad x \geqslant \frac{\gamma_\mu}{\gamma_\nu \sqrt{C_{\mu,\nu}}}.$$

This allows us to deduce from Theorem 1 the following simpler statement with a somewhat weaker conclusion.

COROLLARY 1. *In the situation of Theorem* 1, *the polynomial f has at least k zeros in the strip*

$$\left\{ z \in \mathbb{C} : |\Im z| \leqslant \left(\sqrt{\sum_{\nu=0}^{k-2} \frac{\gamma_\nu}{\gamma_{k-2}} \left| \frac{a_\nu}{a_n} \right|^2} - \frac{\gamma_m}{\gamma_{k-2} \sqrt{C_{m,k-2}}} \right)^{1/(n-k+2)} \right\}$$

provided that the term in parentheses is non-negative, else f has at least k distinct real zeros of odd multiplicities.

For $k = n$, Corollary 1 reduces essentially to Theorem B. The missing separation property is obtained by a continuity argument.

If a bound for $C_{m,k-2}$ is not available, then we may replace in Corollary 1 the whole fraction containing $C_{m,k-2}$ by zero. We still obtain a non-trivial result, which may be attributed to Specht who proved it by a different method in an unpublished manuscript.

Turán established a sufficient condition for an Hermite expansion to have only real zeros. Denoting by H_n^* the nth monic Hermite polynomial, we have for that orthogonal system

$$H_n^*(z) = 2^{-n} H_n(z), \qquad \gamma_n = \frac{n!}{2^n}, \quad C_{n,n-1} = n \qquad (n \in \mathbb{N}). \tag{13}$$

Now Corollary 1 with $k = n$ implies the following criterion, which is exactly that of Turán [19, Theorem III].

COROLLARY 2. *If the Hermite expansion*

$$f(z) = \sum_{\nu=0}^{n} b_\nu H_\nu(z)$$

of a polynomial f has real coefficients satisfying

$$\sum_{\nu=0}^{n-2} 2^\nu \nu!\, b_\nu^2 < 2^n (n-1)!\, b_n^2,$$

then f has n distinct real zeros.

The criteria for real zeros contained in Corollaries 1 and 2 are again sharp.

Let us now consider a quadratic majorant for $K_{\mu,\nu}$. From the definition in (11), it is geometrically evident (cf. Fig. 1) that

$$K_{\mu,\nu}(x) \leqslant \frac{\gamma_\nu}{4\gamma_\mu} x^2 \qquad \text{for} \quad x \geqslant 0.$$

This allows us to deduce from Theorem 1 the following simpler but somewhat weaker statement.

COROLLARY 3. *In the situation of Theorem* 1, *the polynomial f has at least k zeros in the strip*

$$\left\{ z \in \mathbb{C} : |\Im z| \leqslant \left(\frac{1}{4} \sum_{\nu=0}^{k-2} \frac{\gamma_\nu}{\gamma_m} \left| \frac{a_\nu}{a_n} \right|^2 \right)^{1/(n-k+2)} \right\}.$$

For $k = n$, Corollary 3 shows that the moduli of the imaginary parts of the zeros of f are bounded by

$$\frac{1}{2} \sqrt{\sum_{\nu=0}^{n-2} \frac{\gamma_\nu}{\gamma_{n-1}} \left| \frac{a_\nu}{a_n} \right|^2} \tag{14}$$

(also see [15, Satz 1*] and [9, Section 7], where this bound was deduced in different ways). Comparing with (6), we see that Corollary 3, and so Theorem 1 all the more, give a better bound for the imaginary parts of the zeros than Theorem A. Note that (14) cannot yield anything for the real parts of the zeros since it does not contain the coefficient a_{n-1}, but as $a_{n-1} \to \infty$, one of the zeros will approach infinity.

Our method allows us to establish an L^2 inequality for polynomials, which may be of independent interest.

THEOREM 2. *Let σ be an m-distribution. Suppose that f and g are monic polynomials of degree n and k, respectively. Then*

$$\frac{\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}\sigma(x)}{\int_{-\infty}^{\infty} |f(x)\, g(x)|^2 \, \mathrm{d}\sigma(x)} \leqslant \sum_{\nu=0}^{n} \frac{\gamma_\nu}{\gamma_{\nu+k}} C_{\nu+k,\nu},$$

where the numbers γ_ν and $C_{\nu+k,\nu}$ are given by the orthogonal polynomials with respect to $\mathrm{d}\sigma(x)$ *as in* (4) *and* (7).

In Theorem 2, the roles of f and g may be interchanged. Doing so and multiplying the resulting inequality with the previous one, we obtain the following result.

COROLLARY 4. *Let σ be an m-distribution defining the norm* (3). *Suppose that f and g are monic polynomials of degree n and k, respectively. Then*

$$\|fg\|^2 \geqslant M_{n,k}(\sigma)\, \|f\| \cdot \|g\|$$

with a positive number $M_{n,k}(\sigma)$ which depends only on n, k, and σ. In terms of the quantities (4) *and* (7), *that number may be expressed as*

$$M_{n,k}(\sigma) = \left[\left(\sum_{\nu=0}^{n} \frac{\gamma_\nu}{\gamma_{\nu+k}}\, C_{\nu+k,\nu} \right) \left(\sum_{\mu=0}^{k} \frac{\gamma_\mu}{\gamma_{\mu+n}}\, C_{\mu+n,\mu} \right) \right]^{-1/2}.$$

EXAMPLE. For $d\sigma(x) = e^{-x^2}\, dx$, the associated orthogonal polynomials are the Hermite polynomials. With the constants given in (13), we find for $k = 1$ that

$$\frac{\int_{-\infty}^{\infty} |f(x)|^2\, e^{-x^2}\, dx}{\int_{-\infty}^{\infty} |f(x)(x-\zeta)|^2\, e^{-x^2}\, dx} \leqslant 2(n+1)$$

for every monic polynomial f of degree n and all $\zeta \in \mathbb{C}$. When $k > 1$, we can always avoid the appearance of the numbers $C_{\nu+k,\nu}$ by iterating the estimate for $k = 1$ in an obvious way. In our example, we obtain that

$$\frac{\int_{-\infty}^{\infty} |f(x)|^2\, e^{-x^2}\, dx}{\int_{-\infty}^{\infty} |f(x)\, g(x)|^2\, e^{-x^2}\, dx} \leqslant 2^k (n+1) \cdots (n+k)$$

for every pair of monic polynomials f and g of degree n and k, respectively.

2. LEMMAS

Let σ be an m-distribution and $\{\phi_n\}_{n \in \mathbb{N}_0}$ be the sequence of monic orthogonal polynomials with respect to $d\sigma(x)$, as introduced in the previous section. Given a polynomial

$$\omega(z) := \prod_{j=1}^{\ell} (z - \zeta_j)(z - \bar{\xi}_j),$$

we denote by

$$\phi_0^{[\omega]}(z),\ \phi_1^{[\omega]}(z),\ \ldots,\ \phi_n^{[\omega]}(z),\ \ldots$$

the monic orthogonal polynomials with respect to $\omega(x)\, d\sigma(x)$ and define

$$\gamma_n^{[\omega]} := \int_{-\infty}^{\infty} (\phi_n^{[\omega]}(x))^2\, \omega(x)\, d\sigma(x) \qquad (n \in \mathbb{N}_0).$$

Connections between the two sequences $\{\phi_n\}$ and $\{\phi_n^{[\omega]}\}$ have been studied since a long time. In 1858, Christoffel expressed $\phi_n^{[\omega]}$ by a determinant in terms of ϕ_n, ϕ_{n+1}, ..., $\phi_{n+2\ell}$ (see [16, Section 2.5]). A more recent contribution emphasizing the computational aspects is given in [11]. In our approach, we shall need bounds for $\|\phi_n^{[\omega]}\|$ and $\gamma_n^{[\omega]}$. First we mention two useful observations, stating them as lemmas or convenient reference.

LEMMA 1. *Let ψ_k be a monic divisor of the nth orthogonal polynomial ϕ_n with* $\deg \psi_k = k$, *and set* $\omega := (\phi_n/\psi_k)^2$. *Then* $\phi_k^{[\omega]} = \psi_k$.

Proof. It is known (e.g., [16, p. 39, Theorem 3.1.2]) that orthogonal polynomials have the following extremal property. Among all monic polynomials f of degree n, the orthogonal polynomial ϕ_n is the only one which minimizes $\|f\|^2$.

If g is an arbitrary monic polynomial of degree k, then $|g(x)|^2\,\omega(x)$ is the square of the modulus of a monic polynomials of degree n evaluated at $x \in \mathbb{R}$. Hence

$$\int_{-\infty}^{\infty} |g(x)|^2\,\omega(x)\,\mathrm{d}\sigma(x) \geqslant \int_{-\infty}^{\infty} (\phi_n(x))^2\,\mathrm{d}\sigma(x)$$

$$= \int_{-\infty}^{\infty} (\psi_k(x))^2\,\omega(x)\,\mathrm{d}\sigma(x).$$

Now the analogous extremal property for the orthogonal polynomials with respect to $\omega(x)\,\mathrm{d}\sigma(x)$ implies that $\psi_k = \phi_k^{[\omega]}$. ∎

LEMMA 2. *Let* $\omega(z) = \prod_{j=1}^{\ell} (z - \zeta_j)(z - \bar{\zeta}_j)$ *with* $\ell \geqslant 1$, *and let ψ be a polynomial of degree n with real coefficients. Then there exists a polynomial h of degree at most n with real coefficients so that ωh has an expansion*

$$\omega(z)\,h(z) = \sum_{\nu=0}^{n+2\ell} a_\nu \phi_\nu(z)$$

with

$$\sum_{\nu=0}^{n} a_\nu \phi_\nu(z) = \psi(z).$$

Proof. For arbitrary real numbers b_0, ..., b_n, we consider the polynomial

$$g(z) := \omega(z) \sum_{\nu=0}^{n} b_\nu \phi_\nu^{[\omega]}(z)$$

and expand it as

$$g(z)=\sum_{\nu=0}^{n+2\ell} a_\nu \phi_\nu(z). \tag{15}$$

The coefficients $b_0, ..., b_n$ can be expressed in terms of $a_0, ..., a_n$ as follows:

$$\begin{aligned}\gamma_\nu^{[\omega]} b_\nu &= \int_{-\infty}^{\infty} g(x)\, \phi_\nu^{[\omega]}(x)\, \mathrm{d}\sigma(x)\\ &= \sum_{\mu=0}^{n+2\ell} a_\mu \int_{-\infty}^{\infty} \phi_\mu(x)\, \phi_\nu^{[\omega]}(x)\, \mathrm{d}\sigma(x)\\ &= \sum_{\mu=0}^{\nu} a_\mu \int_{-\infty}^{\infty} \phi_\mu(x)\, \phi_\nu^{[\omega]}(x)\, \mathrm{d}\sigma(x) \qquad (\nu=0, ..., n).\end{aligned}$$

This shows that $b_0, ..., b_n$ can be chosen such that $a_0, ..., a_n$ take prescribed values. In particular, there exists a polynomial

$$h(z)=\sum_{\nu=0}^{n} b_\nu \phi_\nu^{[\omega]}(z) \qquad (b_0, ..., b_n \in \mathbb{R})$$

so that $g := \omega h$ has the form (15) with $\sum_{\nu=0}^{n} a_\nu \phi_\nu(z)=\psi(z)$. This completes the proof. ▮

LEMMA 3. *Let* $\omega(z)=\prod_{j=1}^{\ell} (z-\zeta_j)(z-\bar{\zeta}_j)$ *with* $\ell \geqslant 1$. *Then*

$$\gamma_n \leqslant \|\phi_n^{[\omega]}\|^2 \leqslant \gamma_n C_{n+\ell,n} \qquad (n \in \mathbb{N}_0) \tag{16}$$

with $\|\cdot\|$, γ_n, *and* $C_{n+\ell,n}$ *defined by* (3), (4), *and* (7). *These inequalities are best possible.*

Proof. The before-mentioned extremal property of orthogonal polynomials shows immediately that the first inequality in (16) holds.

Now we turn to the second inequality. By Lemma 2 with n replaced by $n-1$, there exists a polynomial h with real coefficients and degree at most $n-1$ so that

$$g(z) := \omega(z)\, h(z)=\sum_{\nu=0}^{n+2\ell-1} a_\nu \phi_\nu(z) \tag{17}$$

with

$$\sum_{\nu=0}^{n-1} a_\nu \phi_\nu(z)=\phi_n^{[\omega]}(z)-\phi_n(z).$$

Then

$$\sum_{\nu=0}^{n-1} \gamma_\nu a_\nu^2 = \|\phi_n^{[\omega]} - \phi_n\|^2. \tag{18}$$

Moreover, since h is of degree at most $n-1$, we have

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} h(x)\, \phi_n^{[\omega]}(x)\, \omega(x)\, \mathrm{d}\sigma(x) = \int_{-\infty}^{\infty} g(x)\, \phi_n^{[\omega]}(x)\, \mathrm{d}\sigma(x) \\ &= \sum_{\nu=0}^{n} a_\nu \int_{-\infty}^{\infty} \phi_\nu(x)\, \phi_n^{[\omega]}(x)\, \mathrm{d}\sigma(x), \end{aligned}$$

and so

$$\begin{aligned} \gamma_n^2 a_n^2 &= \left(\sum_{\nu=0}^{n-1} a_\nu \int_{-\infty}^{\infty} \phi_\nu(x)\, \phi_n^{[\omega]}(x)\, \mathrm{d}\sigma(x) \right)^2 \\ &= \left(\sum_{\nu=0}^{n-1} \gamma_\nu a_\nu^2 \right)^2 = \|\phi_n^{[\omega]} - \phi_n\|^4. \end{aligned} \tag{19}$$

As (17) shows, g has at most $n-1$ real zeros of odd multiplicities. Hence Theorem C implies that

$$\gamma_n a_n^2 \leqslant (C_{n+\ell,n} - 1) \sum_{\nu=0}^{n-1} \gamma_\nu a_\nu^2.$$

By (18) and (19), this is equivalent to

$$\frac{1}{\gamma_n} \|\phi_n^{[\omega]} - \phi_n\|^4 \leqslant (C_{n+\ell,n} - 1)\, \|\phi_n^{[\omega]} - \phi_n\|^2,$$

which yields that

$$\|\phi_n^{[\omega]} - \phi_n\|^2 + \gamma_n \leqslant \gamma_n C_{n+\ell,n}.$$

Observing that the left-hand side is equal to $\|\phi_n^{[\omega]}\|^2$, we arrive at the second inequality in (16).

It remains to show that inequalities (16) cannot be improved.

Let $\xi_1, ..., \xi_{n+\ell}$ be the zeros of $\phi_{n+\ell}$. According to the theory of Gaussian quadrature formulae, there exist positive numbers $\lambda_1, ..., \lambda_{n+\ell}$ so that the equation

$$\int_{-\infty}^{\infty} f(x)\, \mathrm{d}\sigma(x) = \sum_{j=1}^{n+\ell} \lambda_j f(\xi_j) \tag{20}$$

holds for all polynomials f up to degree $2n+2\ell-1$. Now let

$$\omega(z) := (z^2+\eta^2)^\ell \qquad (\eta > 0).$$

Keeping in mind that among all monic polynomials f of degree n, the polynomial $\phi_n^{[\omega]}$ is the only one which minimizes the integral

$$\int_{-\infty}^{\infty} |f(x)|^2\, \omega(x)\, \mathrm{d}\sigma(x),$$

and noting that $(\phi_n^{[\omega]}(x))^2$,

$$(\phi_n^{[\omega]}(x))^2\, \omega(x) - (\phi_{n+\ell}(x))^2 \qquad \text{and} \qquad (\phi_n(x))^2\, \omega(x) - (\phi_{n+\ell}(x))^2$$

are polynomials of degrees not exceeding $2n+2\ell-1$, we may use the quadrature formula (20) to argue as follows:

$$\begin{aligned}
\|\phi_n^{[\omega]}\|^2 &= \sum_{j=1}^{n+\ell} \lambda_j (\phi_n^{[\omega]}(\xi_j))^2 \\
&= \sum_{j=1}^{n+\ell} \lambda_j \frac{(\phi_n^{[\omega]}(\xi_j))^2}{\omega(\xi_j)}\, \omega(\xi_j) \\
&\leqslant \eta^{-2\ell} \sum_{j=1}^{n+\ell} \lambda_j (\phi_n^{[\omega]}(\xi_j))^2\, \omega(\xi_j) \\
&= \eta^{-2\ell} \sum_{j=1}^{n+\ell} \lambda_j [(\phi_n^{[\omega]}(\xi_j))^2\, \omega(\xi_j) - (\phi_{n+\ell}(\xi_j))^2] \\
&= \eta^{-2\ell} \int_{-\infty}^{\infty} [(\phi_n^{[\omega]}(x))^2\, \omega(x) - (\phi_{n+\ell}(x))^2]\, \mathrm{d}\sigma(x) \\
&\leqslant \eta^{-2\ell} \int_{-\infty}^{\infty} [(\phi_n(x))^2\, \omega(x) - (\phi_{n+\ell}(x))^2]\, \mathrm{d}\sigma(x) \\
&= \eta^{-2\ell} \sum_{j=1}^{n+\ell} \lambda_j (\phi_n(\xi_j))^2\, \omega(\xi_j) \\
&\leqslant \eta^{-2\ell} \max_{1 \leqslant j \leqslant n+\ell} (\xi_j^2+\eta^2)^\ell \sum_{j=1}^{n+\ell} \lambda_j (\phi_n(\xi_j))^2 \\
&= \max_{1 \leqslant j \leqslant n+\ell} \left(1+\frac{\xi_j^2}{\eta^2}\right)^\ell \int_{-\infty}^{\infty} (\phi_n(x))^2\, \mathrm{d}\sigma(x) \\
&= \max_{1 \leqslant j \leqslant n+\ell} \left(1+\frac{\xi_j^2}{\eta^2}\right)^\ell \gamma_n.
\end{aligned}$$

The last term approaches γ_n as $\eta \to \infty$. This shows that, for each $n \in \mathbb{N}_0$, the two sides of the first inequality in (16) can be arbitrarily close. For certain integers n even equality may occur. Of course, $\phi_0(x) \equiv \phi_0^{[\omega]}(x) \equiv 1$, and so equality occurs for $n=0$. Furthermore, if $d\sigma(x)$ and $\omega(x)\, d\sigma(x)$ are symmetric with respect to the origin, then $\phi_1(x) \equiv \phi_1^{[\omega]}(x) \equiv x$, which gives equality for $n=1$.

It is easy to see that equality can be attained in the second inequality in (16). In fact, from (7) it is clear that there exists a monic divisor ψ_n of $\phi_{n+\ell}$ with $\deg \psi_n = n$ and

$$\|\psi_n\|^2 = \gamma_n C_{n+\ell,\, n}.$$

But by Lemma 1, $\psi_n = \phi_n^{[\omega]}$ if ω is taken as $(\phi_{n+\ell}/\psi_n)^2$ which is an admissible choice. This completes the proof. ▮

Remark. The proof of Lemma 3 reveals that, if in Theorem C the constant $C_{m,k}$ could be replaced by a smaller number, then the second inequality in (16) could be improved accordingly. But (16) is best possible as we have shown. Hence the constant $C_{m,k}$ in Theorem C must be best possible.

LEMMA 4. *Let $\omega(z) = \prod_{j=1}^{\ell} (z-\zeta_j)(z-\bar{\zeta}_j)$ with $\ell \geqslant 1$. Suppose that*

$$d_{n+\ell}(\zeta_j) \geqslant r \geqslant 0 \qquad (j=1, ..., \ell).$$

Then

$$\gamma_n^{[\omega]} \geqslant \gamma_{n+\ell} + \|\phi_n^{[\omega]}\|^2\, r^{2\ell} \qquad (n \in \mathbb{N}_0). \tag{21}$$

Proof. For $r=0$, the above-mentioned extremal property of orthogonal polynomials, used for $\phi_{n+\ell}$, shows immediately that (21) holds.

Let us now suppose that $r>0$. We shall use again the Gaussian quadrature formula (20), which holds for all polynomials f up to degree $2n+2\ell-1$. Keeping in mind that the nodes of (20) are the zeros of $\phi_{n+\ell}$ and that the coefficients are positive, and noting that $(\phi_n^{[\omega]}(x))^2$ and

$$(\phi_n^{[\omega]}(x))^2\, \omega(x) - (\phi_{n+\ell}(x))^2$$

are polynomials of degrees not exceeding $2n+2\ell-1$, we may proceed as follows:

$$
\begin{aligned}
\|\phi_n^{[\omega]}\|^2 &= \int_{-\infty}^{\infty} (\phi_n^{[\omega]}(x))^2 \, \mathrm{d}\sigma(x) \\
&= \sum_{j=1}^{n+\ell} \lambda_j (\phi_n^{[\omega]}(\xi_j))^2 \\
&= \sum_{j=1}^{n+\ell} \lambda_j \frac{(\phi_n^{[\omega]}(\xi_j))^2}{\omega(\xi_j)} \, \omega(\xi_j) \\
&\leqslant \frac{1}{(d_{n+\ell}(\zeta_1) \cdots d_{n+\ell}(\zeta_\ell))^2} \sum_{j=1}^{n+\ell} \lambda_j (\phi_n^{[\omega]}(\xi_j))^2 \, \omega(\xi_j) \\
&\leqslant r^{-2\ell} \sum_{j=1}^{n+\ell} \lambda_j [(\phi_n^{[\omega]}(\xi_j))^2 \, \omega(\xi_j) - (\phi_{n+\ell}(\xi_j))^2] \\
&= r^{-2\ell} \int_{-\infty}^{\infty} [(\phi_n^{[\omega]}(x))^2 \, \omega(x) - (\phi_{n+\ell}(x))^2] \, \mathrm{d}\sigma(x) \\
&= r^{-2\ell} (\gamma_n^{[\omega]} - \gamma_{n+\ell}).
\end{aligned}
$$

This shows that (21) holds. ▮

Since the function $K_{\mu,\nu}$ as defined in (11) is monotonically increasing and continuously differentiable, it has an inverse $\Lambda_{\mu,\nu} := K_{\mu,\nu}^{-1}$ which is again monotonically increasing and continuously differentiable. By a simple calculation, we find that

$$
\Lambda_{\mu,\nu}(x) = \begin{cases} \sqrt{C_{\mu,\nu}} \left(x + \dfrac{\gamma_\mu}{\gamma_\nu C_{\mu,\nu}} \right) & \text{if } \; -\dfrac{\gamma_\mu}{\gamma_\nu C_{\mu,\nu}} \leqslant x < \dfrac{\gamma_\mu}{\gamma_\nu C_{\mu,\nu}} \\ 2 \sqrt{\dfrac{\gamma_\mu x}{\gamma_\nu}} & \text{if } \; \dfrac{\gamma_\mu}{\gamma_\nu C_{\mu,\nu}} \leqslant x < \dfrac{\gamma_\mu}{\gamma_\nu} \\ x + \dfrac{\gamma_\mu}{\gamma_\nu} & \text{if } \; x \geqslant \dfrac{\gamma_\mu}{\gamma_\nu}. \end{cases} \tag{22}
$$

LEMMA 5. *Let* $\omega(z) = \prod_{j=1}^{\ell} (z - \zeta_j)(z - \bar{\zeta}_j)$ *with* $\ell \geqslant 1$. *Suppose that*

$$
d_{n+\ell}(\zeta_j) \geqslant r \geqslant 0 \qquad (j = 1, ..., \ell).
$$

Then

$$
\frac{\gamma_n^{[\omega]}}{\sqrt{\gamma_n} \, \|\phi_n^{[\omega]}\|} \geqslant \Lambda_{n+\ell,n}(r^{2\ell}) \tag{23}
$$

with the function $\Lambda_{n+\ell,n}$ *given by* (22).

Proof. Introducing the function

$$\varphi\colon t \mapsto \frac{1}{\sqrt{\gamma_n}}\left(\frac{\gamma_{n+\ell}}{t} + r^{2\ell} t\right) \qquad (t>0),$$

we find with the help of Lemma 4 that

$$\frac{\gamma_n^{[\omega]}}{\sqrt{\gamma_n}\,\|\phi_n^{[\omega]}\|} \geqslant \varphi(\|\phi_n^{[\omega]}\|).$$

We shall show that

$$\varphi(\|\phi_n^{[\omega]}\|) \geqslant \Lambda_{n+\ell,n}(r^{2\ell}).$$

For $r=0$ the function φ is decreasing, and so by Lemma 3,

$$\varphi(\|\phi_n^{[\omega]}\|) \geqslant \varphi(\sqrt{\gamma_n C_{n+\ell,n}}) = \Lambda_{n+\ell,n}(0).$$

For $r\neq 0$ the function φ is decreasing on $(0, \sqrt{\gamma_{n+\ell}}\,r^{-\ell})$ and increasing on $(\sqrt{\gamma_{n+\ell}}\,r^{-\ell}, +\infty)$. Hence by Lemma 3, if

$$(\sqrt{\gamma_n}, \sqrt{\gamma_n C_{n+\ell,n}}) \subseteq (0, \sqrt{\gamma_{n+\ell}}\,r^{-\ell}),$$

or equivalently,

$$r^{2\ell} \leqslant \frac{\gamma_{n+\ell}}{\gamma_n C_{n+\ell,n}},$$

then

$$\varphi(\|\phi_n^{[\omega]}\|) \geqslant \varphi(\sqrt{\gamma_n C_{n+\ell,n}}) = \Lambda_{n+\ell,n}(r^{2\ell}).$$

Again by Lemma 3, if

$$(\sqrt{\gamma_n}, \sqrt{\gamma_n C_{n+\ell,n}}) \subseteq (\sqrt{\gamma_{n+\ell}}\,r^{-\ell}, +\infty),$$

or equivalently,

$$r^{2\ell} \geqslant \frac{\gamma_{n+\ell}}{\gamma_n},$$

then

$$\varphi(\|\phi_n^{[\omega]}\|) \geqslant \varphi(\sqrt{\gamma_n}) = \Lambda_{n+\ell,n}(r^{2\ell}).$$

It remains to establish an estimate for

$$\frac{\gamma_{n+\ell}}{\gamma_n C_{n+\ell,n}} < r^{2\ell} < \frac{\gamma_{n+\ell}}{\gamma_n}. \tag{24}$$

But for every $r > 0$, the function φ has an absolute minimum at $\sqrt{\gamma_{n+\ell}}\, r^{-\ell}$. Hence

$$\varphi(\|\phi_n^{[\omega]}\|) \geqslant \varphi(\sqrt{\gamma_{n+\ell}}\, r^{-\ell}).$$

The right-hand side is equal to $\Lambda_{n+\ell,n}(r^{2\ell})$ if $r^{2\ell}$ is restricted by (24). This completes the proof. ∎

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. In the situation of Theorem 1, set

$$\ell := \frac{n-k}{2} + 1,$$

and assume that

$$f(z) = \sum_{\nu=0}^{n} a_\nu \phi_\nu(z)$$

has ℓ pairs of conjugate zeros ζ_j, $\bar{\zeta}_j$ $(j = 1, ..., \ell)$ satisfying

$$d_m(\zeta_1) \geqslant d_m(\zeta_2) \geqslant \cdots \geqslant d_m(\zeta_\ell) =: r, \tag{25}$$

while $d_m(\zeta) \leqslant r$ for any other zero ζ of f. Setting

$$\omega(z) := \prod_{j=1}^{\ell} (z - \zeta_j)(z - \bar{\zeta}_j),$$

we have a factorization

$$f(z) = \omega(z)\; g(z),$$

where g is a polynomial of degree $n - 2\ell$, which may be represented as

$$g(z) = \sum_{\nu=0}^{n-2\ell} b_\nu \phi_\nu^{[\omega]}(z).$$

Obviously $b_{n-2\ell}=a_n$, and so, using the orthogonality property, we find that

$$\begin{aligned}a_n\gamma^{[\omega]}_{n-2\ell} &= b_{n-2\ell}\int_{-\infty}^{\infty}(\phi^{[\omega]}_{n-2\ell}(x))^2\,\omega(x)\,\mathrm{d}\sigma(x)\\ &= \int_{-\infty}^{\infty} g(x)\,\phi^{[\omega]}_{n-2\ell}(x)\,\omega(x)\,\mathrm{d}\sigma(x)\\ &= \int_{-\infty}^{\infty} f(x)\,\phi^{[\omega]}_{n-2\ell}(x)\,\mathrm{d}\sigma(x)\\ &= \sum_{\nu=0}^{n-2\ell} a_\nu \int_{-\infty}^{\infty}\phi_\nu(x)\,\phi^{[\omega]}_{n-2\ell}(x)\,\mathrm{d}\sigma(x).\end{aligned}$$

Estimating the right-hand side with the help of the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\gamma^{[\omega]}_{n-2\ell} &\leqslant \sqrt{\sum_{\nu=0}^{n-2\ell}\gamma_\nu\left|\frac{a_\nu}{a_n}\right|^2}\cdot\sqrt{\sum_{\nu=0}^{n-2\ell}\frac{1}{\gamma_\nu}\left(\int_{-\infty}^{\infty}\phi_\nu(x)\,\phi^{[\omega]}_{n-2\ell}(x)\,\mathrm{d}\sigma(x)\right)^2}\\ &= \sqrt{\sum_{\nu=0}^{n-2\ell}\gamma_\nu\left|\frac{a_\nu}{a_n}\right|^2}\cdot\|\phi^{[\omega]}_{n-2\ell}\|.\end{aligned}$$

Hence

$$\frac{\gamma^{[\omega]}_{n-2\ell}}{\sqrt{\gamma_{n-2\ell}}\;\|\phi^{[\omega]}_{n-2\ell}\|}\leqslant\sqrt{\sum_{\nu=0}^{n-2\ell}\frac{\gamma_\nu}{\gamma_{n-2\ell}}\left|\frac{a_\nu}{a_n}\right|^2}.$$

Employing Lemma 5 with n replaced by $n-2\ell$ and keeping in mind that

$$n-\ell=m \qquad\text{and}\qquad n-2\ell=k-2,$$

we deduce that

$$\Lambda_{m,k-2}(r^{2\ell})\leqslant\sqrt{\sum_{\nu=0}^{k-2}\frac{\gamma_\nu}{\gamma_{k-2}}\left|\frac{a_\nu}{a_n}\right|^2}=:M \tag{26}$$

with the function $\Lambda_{m,k-2}$ given by (22).

Now we argue as follows. If $M<\Lambda_{m,k-2}(0)$, or equivalently,

$$\sum_{\nu=0}^{k-2}\gamma_\nu a_\nu^2<\frac{\gamma_m^2}{\gamma_{k-2}C_{m,k-2}}\,a_n^2,$$

then (26) cannot be satisfied with an $r \geqslant 0$. This means that the assumption (25) cannot hold. Hence f has at most $\ell - 1$ pairs of conjugate zeros including real double zeros. It has to be noted that a real zero ζ of multiplicity $2\kappa + 1$ counts as κ pairs of conjugate zeros and one further zero which all coalesce at ξ. Thus, if the zeros of f consist of λ pairs of non-real conjugate zeros, μ distinct real zeros of multiplicities $2m_1 + 1, ..., 2m_\mu + 1$, and v distinct real zeros of multiplicities $2n_1, ..., 2n_v$, then

$$2(\lambda + m_1 + \cdots + m_\mu + n_1 + \cdots + n_v) + \mu = n$$

and

$$\lambda + m_1 + \cdots + m_\mu + n_1 + \cdots + n_v \leqslant \ell - 1.$$

This implies that

$$\mu \geqslant n - 2\ell + 2 = k,$$

that is, f has at least k distinct real zeros of odd multiplicities.

If $M \geqslant \Lambda_{m,k-2}(0)$, then there exist numbers $r \geqslant 0$ so that (26) is satisfied. Recalling that the functions $K_{\mu,v}$ and $\Lambda_{\mu,v}$ as defined in (11) and (22) are both monotonically increasing and $K_{\mu,v}$ is the inverse of $\Lambda_{\mu,v}$, we deduce from (26) that

$$r \leqslant (K_{m,k-2}(M))^{1/(2\ell)}.$$

In view of (25), we see that at most $\ell - 1$ pairs of conjugate zeros can lie outside

$$\{z \in \mathbb{C} : d_m(z) \leqslant (K_{m,k-2}(M))^{1/(2\ell)}\}.$$

It remains to show that in the condition (12) the constant $C_{m,k-2}$ is best possible. Assume to the contrary that it can be replaced by a number

$$\tilde{C}_{m,k-2} < C_{m,k-2}.$$

Let ψ_{k-2} be a monic divisor of ϕ_m with $\deg \psi_{k-2} = k-2$ such that

$$\|\psi_{k-2}\|^2 = \gamma_{k-2} C_{m,k-2}.$$

Setting $\omega := (\phi_m/\psi_{k-2})^2$, we know from Lemma 1 that $\psi_{k-2} = \phi_{k-2}^{[\omega]}$. Moreover, by Lemma 2 there exists a polynomial h of degree at most $k-2$ with real coefficients so that $f := \omega h$ has an expansion

$$f(z) = \omega(z)\, h(z) = \sum_{v=0}^{n} a_v \phi_v(z) \tag{27}$$

with

$$\sum_{\nu=0}^{k-2} a_\nu \phi_\nu(z) = \psi_{k-2}(z).$$

Therefore

$$\sum_{\nu=0}^{k-2} \gamma_\nu a_\nu^2 = \|\psi_{k-2}\|^2 = \gamma_{k-2} C_{m,k-2}. \tag{28}$$

Furthermore, expanding h as

$$h(z) = \sum_{j=0}^{k-2} b_j \phi_j^{[\omega]}(z)$$

and noting that $b_{k-2} = a_n$, we find that

$$\begin{aligned}
\gamma_{k-2}^{[\omega]} a_n &= \gamma_{k-2}^{[\omega]} b_{k-2} = \int_{-\infty}^{\infty} h(x)\, \phi_{k-2}^{[\omega]}(x)\, \omega(x)\, \mathrm{d}\sigma(x) \\
&= \int_{-\infty}^{\infty} f(x)\, \phi_{k-2}^{[\omega]}(x)\, \mathrm{d}\sigma(x) = \sum_{\nu=0}^{k-2} a_\nu \int_{-\infty}^{\infty} \phi_\nu(x)\, \phi_{k-2}^{[\omega]}(x)\, \mathrm{d}\sigma(x) \\
&= \int_{-\infty}^{\infty} (\psi_{k-2}(x))^2\, \mathrm{d}\sigma(x) = \|\psi_{k-2}\|^2 = \gamma_{k-2} C_{m,k-2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\gamma_{k-2}^{[\omega]} &= \int_{-\infty}^{\infty} (\phi_{k-2}^{[\omega]}(x))^2\, \omega(x)\, \mathrm{d}\sigma(x) = \int_{-\infty}^{\infty} (\psi_{k-2}(x))^2\, \omega(x)\, \mathrm{d}\sigma(x) \\
&= \int_{-\infty}^{\infty} (\phi_m(x))^2\, \mathrm{d}\sigma(x) = \gamma_m.
\end{aligned}$$

Combining these equations with the preceding ones, we obtain

$$\gamma_m a_n = \gamma_{k-2} C_{m,k-2}. \tag{29}$$

Now (28) and (29) imply that

$$\sum_{\nu=0}^{k-2} \gamma_\nu a_\nu^2 = \frac{\gamma_m^2}{\gamma_{k-2} C_{m,k-2}}\, a_n^2.$$

Consequently,

$$\sum_{\nu=0}^{k-2} \gamma_\nu a_\nu^2 < \frac{\gamma_m^2}{\gamma_{k-2} \tilde{C}_{m,k-2}} a_n^2.$$

Hence the sharpened form of (12) would be satisfied, but (27) shows that f has at most $k-2$ distinct real zeros of odd multiplicities; a contradiction. This completes the proof. ∎

Proof of Theorem 2. In the situation of Theorem 2, there exists a monic polynomial ω of degree $2k$ with real coefficients such that

$$\omega(x) = |g(x)|^2 \qquad \text{for} \quad x \in \mathbb{R}.$$

Again, we denote by $\{\phi_n^{[\omega]}\}_{n \in \mathbb{N}_0}$ the sequence of monic orthogonal polynomials with respect to $\omega(x)\, d\sigma(x)$ and use all the previous notations. In addition, for arbitrary complex polynomials φ and ψ, we introduce the inner product

$$\langle \varphi, \psi \rangle := \int_{-\infty}^{\infty} \varphi(x)\, \overline{\psi(x)}\, d\sigma(x)$$

and the norm

$$\|\varphi\|_\omega := \left(\int_{-\infty}^{\infty} |\varphi(x)|^2\, \omega(x)\, d\sigma(x) \right)^{1/2}.$$

The polynomial f may be expanded in two ways as

$$f(z) = \sum_{\mu=0}^{n} b_\mu \phi_\mu(z) = \sum_{\nu=0}^{n} b_\nu^{[\omega]} \phi_\nu^{[\omega]}(z).$$

The coefficients $\{b_\mu\}$ and $\{b_\nu^{[\omega]}\}$ are easily seen to be connected by the following equations:

$$\begin{aligned} \gamma_\mu b_\mu = \langle f, \phi_\mu \rangle &= \sum_{\nu=0}^{n} b_\nu^{[\omega]} \langle \phi_\nu^{[\omega]}, \phi_\mu \rangle \\ &= \sum_{\nu=\mu}^{n} b_\nu^{[\omega]} \langle \phi_\nu^{[\omega]}, \phi_\mu \rangle \qquad (\mu = 0, ..., n). \end{aligned} \tag{30}$$

Introducing the vectors

$$\mathbf{x} := \left(\sqrt{\gamma_0^{[\omega]}}\, b_0^{[\omega]}, ..., \sqrt{\gamma_n^{[\omega]}}\, b_n^{[\omega]}\right)^{\mathrm{T}},$$
$$\mathbf{y} := \left(\sqrt{\gamma_0}\, b_0, ..., \sqrt{\gamma_n}\, b_n\right)^{\mathrm{T}},$$

and the triangular matrix

$$C = \begin{pmatrix} c_{00} & c_{01} & \cdots & c_{0n} \\ & c_{11} & \cdots & c_{1n} \\ & & \ddots & \vdots \\ & & & c_{nn} \end{pmatrix}$$

with entries

$$c_{\mu\nu} := \frac{\langle \phi_\mu, \phi_\nu^{[\omega]} \rangle}{\sqrt{\gamma_\mu \gamma_\nu^{[\omega]}}} \qquad (0 \leqslant \mu \leqslant \nu \leqslant n),$$

we may rewrite Eqs. (30) as

$$\mathbf{y} = C\mathbf{x}. \tag{31}$$

Denoting by $|\cdot|$ the Euclidean norm in $\mathbb{C}^{n+1}$ and by $\|C\|$ the spectral norm of the matrix C, we deduce from (31) that

$$|\mathbf{y}| \leqslant \|C\| \cdot |\mathbf{x}|. \tag{32}$$

Moreover,

$$\|f\| = |\mathbf{y}| \qquad \text{and} \qquad \|f\|_\omega = |\mathbf{x}|.$$

Hence (32) may be rewritten as

$$\frac{\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}\sigma(x)}{\int_{-\infty}^{\infty} |f(x)\, g(x)|^2 \, \mathrm{d}\sigma(x)} \leqslant \|C\|^2. \tag{33}$$

Since the spectral norm is bounded from above by the Frobenius norm (e.g., [6, Section 2.3]), we have

$$\|C\|^2 \leqslant \sum_{0 \leqslant \mu \leqslant \nu \leqslant n} c_{\mu\nu}^2 = \sum_{\nu=0}^{n} \sum_{\mu=0}^{\nu} c_{\mu\nu}^2. \tag{34}$$

The last sum may be expressed as

$$\sum_{\mu=0}^{\nu} c_{\mu\nu}^2 = \sum_{\mu=0}^{\nu} \frac{\langle \phi_\mu, \phi_\nu^{[\omega]} \rangle^2}{\gamma_\mu \gamma_\nu^{[\omega]}} = \frac{1}{\gamma_\nu^{[\omega]}} \|\phi_\nu^{[\omega]}\|^2.$$

Estimating the right-hand side with the help of Lemmas 3 and 4, we find that

$$\sum_{\mu=0}^{\nu} c_{\mu\nu}^2 \leqslant \frac{\gamma_\nu}{\gamma_{\nu+k}} C_{\nu+k,\nu}. \tag{35}$$

Now the conclusion of Theorem 2 is obtained by combining (33)–(35). ∎

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